Needham, VDGAF, Chapter 20, Exercise 20, p. 224

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20. Curvature of the n-Saddle*. Given that the height of the n-saddle is* $z = f(r,\ \theta) = r^n \cos n\theta$*, use (20.4) to prove that its (negative) curvature is constant on each circle r = const., and is given by the following formulas.*

(i) The common 2-saddle (surrounding a generic hyperbolic point) has

$$\mathcal{K} = -\frac{4}{\left(1+4\,r^2\right)^2}$$

$$\mathcal{K} = \frac{r^2\,\partial_r^2 f\left(\partial_\theta^2 f + r\,\partial_r f\right) - \left(\partial_\theta f - r\,\partial_r \partial_\theta f\right)^2}{\left\{r^2\left[1+(\partial_r f)^2\right]+(\partial_\theta f)^2\right\}^2} \qquad (20.4)$$

To find the solution for $\mathcal{K}$, take the derivatives of f and substitute into (20.4) using $C = \cos n\theta$, $S = \sin n\theta$.

$$f(r,\ \theta) = r^2 \cos 2\,\theta = r^2\, C$$

$$\partial_r f = 2\,r \cos 2\,\theta = 2\,\mathrm{rC}$$

$$\partial_r^2 f = 2 \cos 2\,\theta = 2\,C$$

$$\partial_\theta f = -2\,r^2 \sin 2\,\theta = -2\,r^2\, S$$

$$\partial_\theta^2 f = -4\,r^2 \cos 2\,\theta = -4\,r^2\, C$$

$$\partial_r \partial_\theta f = -4\,r \sin 2\,\theta = -4\,\mathrm{rS}$$

$$\mathcal{K} = \frac{r^2(2\,C)\left(-4\,r^2\,C + r(2\,\mathrm{rC})\right) - \left[-2\,r^2\,S - r(4\,\mathrm{rS})\right]^2}{\left\{r^2\left[1+(2\,r\,C)^2\right]+\left(-2\,r^2\,S\right)^2\right\}^2}$$

$$= \frac{\left(-8\,r^4\,C^2 + 4\,r^4\,C^2\right) - \left[4\,r^4\,S^2 - 2\times\left(8\,r^4\,S^2\right) + 16\,r^4\,S^2\right]}{\left\{r^2\left(1+4\,r^2\,C^2\right) + 4\,r^4\,S^2\right\}^2}$$

$$= \frac{-4\,r^4\,C^2 - 4\,r^4\,S^2}{\left\{r^2\left(1+4\,r^4\left(C^2+S^2\right)\right)\right\}^2}$$

$$= \frac{-4}{\left(1+4\,r^2\right)^2} \qquad C^2 + S^2 = 1, \text{ both top and bottom}$$

This is what we wanted. $\mathcal{K}$ is constant where r = const.

(ii) The monkey 3-saddle has

$$\mathcal{K} = -\frac{36\, r^2}{\left(1+9\, r^4\right)^2}$$

$$f(r,\ \theta) = r^3 \cos 3\, \theta = r^3\, C$$

$$\partial_r f = 3\, r^2 \cos 3\, \theta = 3\, r^2\, C$$

$$\partial_r^2 f = 6\, r \cos 3\, \theta = 6\, r\, C$$

$$\partial_\theta f = -3\, r^3 \sin 3\, \theta = -3\, r^3\, S$$

$$\partial_\theta^2 f = -9\, r^3 \cos 3\, \theta = -9\, r^3\, C$$

$$\partial_r \partial_\theta f = -9\, r^2 \sin 2\, \theta = -9\, r^2\, S$$

$$\mathcal{K} = \frac{r^2\, 6\, r\, C\left(-9\, r^3\, C + \mathrm{r}3\, r^2\, C\right) - \left(-3\, r^3\, S - r\left(-9\, r^2\, S\right)\right)^2}{\left\{r^2\left[1+\left(3\, r^2\, C\right)^2\right]+\left(-3\, r^3\, S\right)^2\right\}^2} \qquad C^2 + S^2 = 1\text{, both top and bottom}$$

$$= \frac{-36\, r^6}{\left(r^2+9\, r^6\, C^2+9\, r^6\, S^2\right)^2} = -\frac{36\, r^6}{r^2\left(1+9\, r^4\, C^2+9\, r^4\, S^2\right)^2} = -\frac{36\, r^4}{\left(1+9\, r^4\right)^2}$$

This is what we wanted. $\mathcal{K}$ is constant where r = const.

(iii) The n-saddle has

$$\mathcal{K} = -\frac{(n-1)^2\, n^2\, r^{2\,n}}{\left(r^2+n^2\, r^{2\,n}\right)^2}$$

$$f(r,\ \theta) = r^n \cos \mathrm{n}\theta = r^n\, C$$

$$\partial_r f = \mathrm{nr}^{(n-1)} \cos \mathrm{n}\theta = \mathrm{nr}^{(n-1)}\, C$$

$$\partial_r^2 f = (n-1)\, \mathrm{nr}^{(n-2)} \cos \mathrm{n}\theta = (n-1)\, \mathrm{nr}^{(n-2)}\, C$$

$$\partial_\theta f = -\mathrm{nr}^n \sin \mathrm{n}\theta = -\mathrm{nr}^n\, S$$

$$\partial_\theta^2 f = -n^2\, r^n \cos \mathrm{n}\theta = -n^2\, r^n\, C$$

$$\partial_r \partial_\theta f = -n^2\, r^{(n-1)} \sin n\theta = -n^2\, r^{(n-1)}\, S$$

$$\mathcal{K} = \left(r^2 (n-1)\, \mathrm{n} r^{(n-2)}\, C\left(-n^2\, r^n\, C + r\, \mathrm{n} r^{(n-1)}\, C\right) - \left(-\mathrm{n} r^n\, S - r\left(-n^2\, r^{(n-1)}\, S\right)\right)^2\right) \Big/ \left\{r^2\left[1 + \left(\mathrm{n} r^{(n-1)}\, C\right)^2\right] + (-\mathrm{n} r^n\, S)^2\right\}^2$$

$$= \frac{(n-1)\, n\, C\left(-n^2\, r^{2\,n}\, C + \mathrm{n} r^{2\,n}\, C\right) - \left(-\mathrm{n} r^n\, S + n^2\, r^n\, S\right)^2}{\left\{r^2\left[1+\left(\mathrm{n} r^{(n-1)}\, C\right)^2\right] + (-\mathrm{n} r^n\, S)^2\right\}^2}$$

$$= \frac{-(n-1)\, n\left(n^2\, r^{2\,n}\, C^2\right) + (n-1)\left(n^2\, r^{2\,n}\, C^2\right) - \left[n^2\, r^{2\,n}\, S^2 - 2\, n\left(n^2\, r^{2\,n}\, S^2\right) + n^2\left(n^2\, r^{2\,n}\, S^2\right)\right]}{\left\{\left[r^2 + n^2\, r^{2\,n}\, C^2\right] + n^2\, r^{2\,n}\, S^2\right\}^2}$$

$$= \frac{(-(n-1)\, n) + (n-1))\, n^2\, r^{2\,n}\, C^2 - \left(1 - 2\, n + n^2\right) n^2\, r^{2\,n}\, S^2}{\left\{r^2 + n^2\, r^{2\,n}\left(C^2 + S^2\right)\right\}^2}$$

$$= \frac{-(n-1)^2\, n^2\, r^{2\,n}\, C^2 - (n-1)^2\, n^2\, r^{2\,n}\, S^2}{\left\{r^2 + n^2\, r^{2\,n}\right\}^2} = \frac{-(n-1)^2\, n^2\, r^{2\,n}\left(C^2 + S^2\right)}{\left\{r^2 + n^2\, r^{2\,n}\right\}^2}$$

$$= -\frac{(n-1)^2\, n^2\, r^{2\,n}}{\left\{r^2 + n^2\, r^{2\,n}\right\}^2}$$

This is what we wanted. $\mathcal{K}$ is constant where r = const.

In every case, $\mathcal{K}$ is negative and constant where r = const.