Needham, VDGAF, Chapter 20, Exercise 2, p. 219.
Palmer, January 11, 2022, 1:30 pm PT

2. Binormal Cannot Tip in Direction of Motion. *Prove by calculation that as the Frenet frame (**T, N, B**) moves along its curve, the binormal **B** only spins around **T**; it cannot tip in the direction of **T**. (Hint: Symbolically, what must be proved is that **B**' has no component in the **T** direction:* $\boldsymbol{B}'\bullet\boldsymbol{T} = 0$.)

The cross product of two vectors is a third vector orthogonal to the plane of the first two at the point where they intersect. We let $\mathbf{V}'=d\mathbf{V}/ds$. Let a and b be real number coefficients of undetermined value.

$$\boldsymbol{B} = \boldsymbol{T}\times\boldsymbol{N}$$

$$\boldsymbol{B}' = (\boldsymbol{T}\times\boldsymbol{N})' = \boldsymbol{T}'\times\boldsymbol{N} + \boldsymbol{T}\times\boldsymbol{N}'$$

$$\boldsymbol{B}'\bullet\mathbf{T} = (\boldsymbol{T}'\times\boldsymbol{N})\bullet\mathbf{T} + (\boldsymbol{T}\times\boldsymbol{N}')\bullet\mathbf{T}$$

$\boldsymbol{T}'\times\boldsymbol{N} = a\boldsymbol{N}\times\boldsymbol{N} = 0$, so

$$(\boldsymbol{T}'\times\boldsymbol{N})\bullet\mathbf{T} = 0\bullet\mathbf{T} = 0 \qquad (1)$$

Since $\boldsymbol{N}' = -\kappa\boldsymbol{T} + \tau\boldsymbol{B}$,

$$\boldsymbol{T}\times\boldsymbol{N}' = \boldsymbol{T}\times(-\kappa\boldsymbol{T} + \tau\boldsymbol{B}) = -\boldsymbol{T}\times\kappa\boldsymbol{T} + \boldsymbol{T}\times\tau\boldsymbol{B}$$

$$= -\kappa\,\mathbf{T}\times\mathbf{T} + \tau\mathbf{T}\times\mathbf{B} = \tau\mathbf{T}\times\mathbf{B}$$

$\tau\boldsymbol{T}\times\boldsymbol{B}$ points in the direction of $\mathbf{N}$ or $-\boldsymbol{N}$, so

$$(\boldsymbol{T}\times\boldsymbol{N}')\bullet\mathbf{T} = \tau(\boldsymbol{T}\times\boldsymbol{B})\bullet\mathbf{T} = \mathbf{N}\bullet\mathbf{T} = 0 \qquad (2)$$

Therefore, adding (1) and (2), $\boldsymbol{B}'\bullet\boldsymbol{T} = 0$.