Needham, VDGAF, Chapter 20, Exercise 19, p. 224

Palmer, February 4 , 2024, 1:30 pm PDT

19. Nonisometric surfaces with Equal Curvatures. *As we discussed in Act I, Minding proved that if two surfaces have the same constant* $\mathcal{K}$*, then they are locally isometric to each other. Thus, according as this constant curvature* $\mathcal{K} > 0$*,* $\mathcal{K} = 0$*,* $\mathcal{K} < 0$*, the surface is locally isometric to a sphere, plane, or pseudo-sphere, respectively. We now provide an example to show that the converse is false [I.e. that a surface locally isometric to a sphere, plane, or pseudosphere does not prove that the surface has constant curvature* $\mathcal{K} > 0$*,* $\mathcal{K} = 0$*, or* $\mathcal{K} < 0$*].*

Out[∘]=

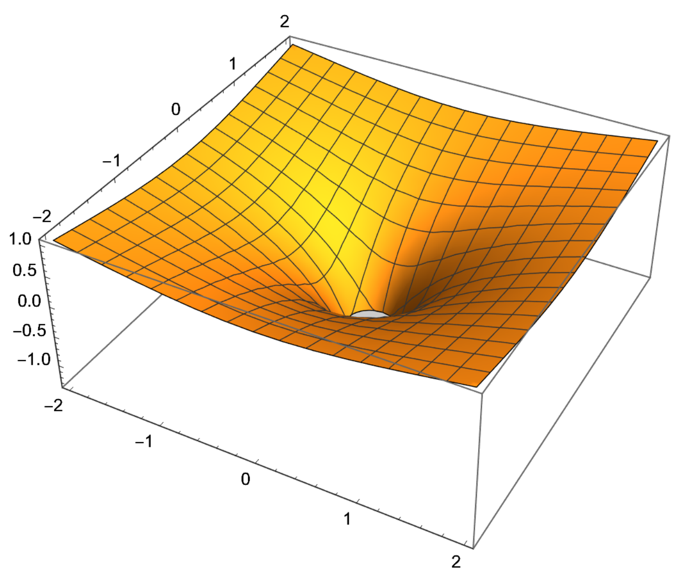


Figure 1. $f_1 = \ln r$, $r = \sqrt{x^2 + y^2}$

Out[∘]=

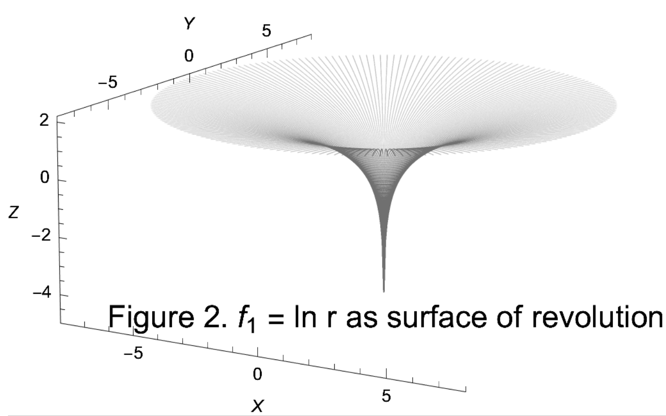


Figure 2. $f_1 = \ln r$ as surface of revolution

(i) Using the same notation as in the previous exercise, find the shapes of these two surfaces: $\mathcal{S}_1$ with $f_1(r,\ \theta) = \ln r$, and $f_2(r,\ \theta) = \theta$. (Hint: $\mathcal{S}_2$ is called the helicoid.) Check your answers by using a computer to draw them.

In § 4.4, ¶¶ 1 and 2, p. 37 Needham stated that if you know the curvature at every point, you can visualize the shape. In both shapes, $\mathcal{K}$ is negative everywhere and depends on r, so rotations have no effect and $f(r,\ \theta)$ has an effect only via r. All points of both shapes with equal r have equal $\mathcal{K}$. Since θ is unspecified in f_1, one can deduce that f_1 ascends by $\ln r$ as r grows, and θ can be anything, so this is a surface of revolution about the Z – axis generating a funnel of infinite radius. Since r is unspecified in f_2, one can deduce that f_2 ascends as θ turns and r can be anything at all (infinite helicoid). So it seems that we are tasked to calculate a formula for $\mathcal{K}$, implicitly in part (i) and explicitly in part (iii). Part (iii) may be redundant, or it may assume that we would not have noticed the equality of $\mathcal{K}$ in $f_1$and f_2 in part (i).

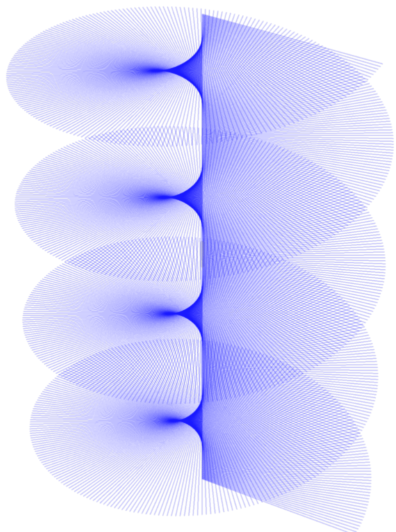
Curvature for f_1 :

$$\partial_r \ln \mathrm{r} = \frac{1}{r}, \qquad \partial_r^2 \ln r = -\frac{1}{r^2}$$

$$\mathcal{K} = \frac{r^2\left(-\frac{1}{r^2}\right)\left(0+r\left(\frac{1}{r}\right)-0^2\right.}{\left\{r^2\left[1+\left(\frac{1}{r}\right)^2+0\right\}^2} = \frac{-1}{r^4\left[1+\left(\frac{1}{r}\right)^2\right]^2} = \frac{-1}{r^4\left[1+2\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^4\right]} = \frac{-1}{\left(r^4+2\,r^2+1\right)} = \frac{-1}{[1+r^2]^2}$$

The curvature is always negative. It approaches 0 when r is very large and −1 when r is very small. It does not depend on θ, so it is symmetric with respect to the z axis and can be seen as a surface of revolution. It must have the shape of a funnel (or petunia, or simple coronet with straight bore) with a brim of infinite radius and spout or bore that extends only down to and including −1. See figures 1 and 2, in which the same function is plotted by two different methods.

Out[∘]=



`Figure 3. Helicoid f₂ = θ`

Curvature for f_2 :

$$\partial_\theta\, \theta = 1, \qquad \partial_\theta^2\, \partial_\theta\, \theta = 0$$

$$\mathcal{K} = \frac{r^2 \cdot 0\,(0+0) - (1-0)}{\left\{r^2[1+0]+1\right\}^2} = \frac{-1}{\left[1+r^2\right]^2}$$

If one looks at only a small portion, it has the appearance of a saddle, which explains negative $\mathcal{K}$. Since r is unspecified, it may be any positive real number out to infinity and, since θ may extend to $\pm\infty$, so may f(r, θ). One can imagine an infinite sequence of horizontal rays plotted from the Z-axis at angle dθ, each rising by dθ, to form a surface of translation spiraling upwards and downwards.

Out[•]=

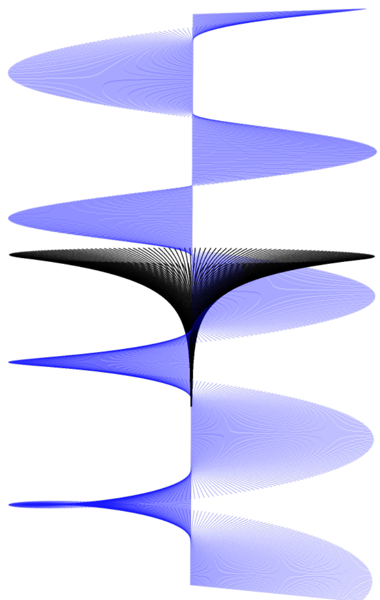


Figure 4a. Superposition of funnel and helicoid

Out[•]=

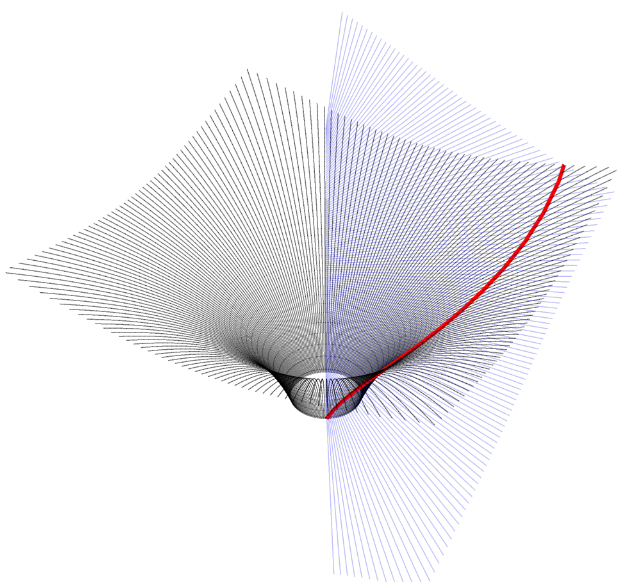


Figure 4b. Superposition of funnel and helicoid
Red line reveals intersection
where metrics of $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal.

(ii) Let us say that a point in S_1 and a point in S_2 "correspond" if they have the same $(r,\ \theta)$ coordinates. By finding the formulas for the metrics of both surfaces, show that this correspondence is not an isometry.

Both Vasco and Tim Bending had the metrics right before I did. I had to go through several revisions before I could see it. Figure 4 shows the helicoid of figure 3 superimposed on the funnel of figure 2. Let these stand in for $\mathcal{S}_1$ (infinite funnel) and $\mathcal{S}_2$ (infinite helicoid). The exercise stipulates only that two points, one from each shape, on the surface above the (r,θ) plane) have the same **(r,$\boldsymbol{\theta}$)** coordinates. This would clearly be the case at every point where the two figures intersect, but it is not clear that they do. Nevertheless, there would still be many points that meet the criterion of correspondence. One need only plot a line parallel to the Z axis and intersecting both shapes so see this.

The correspondences can only be an isometry where the metrics of the two surfaces are equal. To plot the metric in each case, assume a two dimensional surface.

Out[∘]=

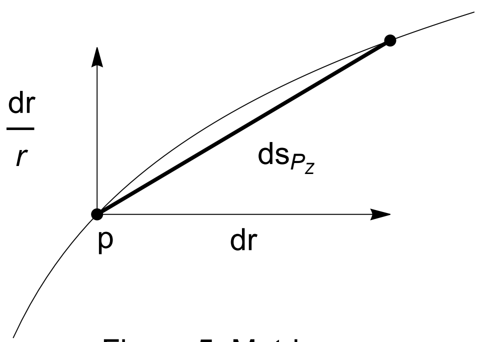


Figure 5. Metric on curve
in plane through Z–axis

Out[∘]=

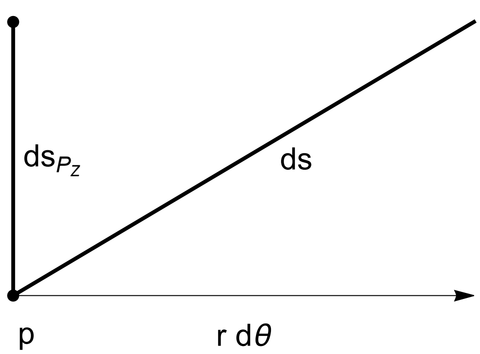


Figure 6. Metric on S_1

Metric for $\boldsymbol{f_1(r,\ \theta) = \ln r}$:
This metric applies to the surface $\mathcal{S}_1$ generated by the curve ln r in a plane P_Z as shown in Figure 5. This plane is a slice through the Z axis and a geodesic on $\mathcal{S}_1$ at any θ angle. The metric for the curve on P_Z is $(\mathrm{ds}_{P_Z})^2 = \left(\mathrm{dr}^2 + \left(\frac{\mathrm{dr}}{r}\right)^2\right)$, which is the square of the hypotenuse of a right triangle with sides $\{\mathrm{dr},\ \frac{\mathrm{dr}}{r}\}$. The infinitesimal chord ds_{P_Z} ultimately lies in the tangent plane of $\mathcal{S}_1$ at any point p on the curve, where it sums with the orthogonal infinitesimal (r dθ) to obtain the metric on $\mathcal{S}_1$ (Figure 6). Since $\partial_r f(r,\ \theta) = \frac{1}{r}$,

the metric on $\mathcal{S}_1$ is

$$ds_1^2 = \left(dr^2 + \frac{dr^2}{r^2}\right) + (r\,d\theta)^2$$

Metric for $f_2(r,\ \theta) = \theta$:
This metric is on the surface of translation of the infinite helicoid, the surface formed by the spiraling of horizontal rays from the Z-axis, shown with a finite radius in Figure 3. One of the orthogonal axes lies along a horizontal ray plotted from the Z-axis with radius r and angle θ when projected to the (r, θ) plane. The other lies in the tangent plane at any point p on a tangent $\partial_\theta f_2$ plotted perpendicular to the ray. The metric along the tangent is $(ds_{\tan})^2 = (\partial_\theta f_2)^2 \cdot d\theta^2 + r^2\,d\theta^2$.

$$\partial_\theta f_2 = 1$$

$$ds_2^2 = dr^2 + \left(1^2 \cdot d\theta^2 + r^2\,d\theta^2\right) = dr^2 + \left(1 + r^2\right) d\theta^2$$

Set $ds_1^2 = ds_2^2$

$$dr^2 + \frac{dr^2}{r^2} + (r\,d\theta)^2 = dr^2 + \left(1 + r^2\right) d\theta^2$$
$$\frac{dr^2}{r^2} = d\theta^2 \qquad \text{(subtracting } dr^2 + r^2\ d\theta^2 \text{ from both sides)}$$
$$d\theta = \frac{dr}{r}$$
$$\theta = \ln r + \text{const.}$$

The funnel and the helicoid are isometric surfaces only where the equation ($\theta = \ln r + \text{const.}$) holds. They are not generally isometric. If we let const. = 0, then $e^\theta = e^{\ln r} = r$. Since r increases by the power of θ, the points of isometry, which are also points of intersection, must spiral outwards as noted by Vasco (P.C. Forum). See Figure 4b. But as θ increases, f_1 and f_2 increase, so the spiral also has a vertical dimension. It is not clear to me that intersection of the surfaces is necessary for equal metrics, but in this case the curves $\left(\mathrm{Re}\left[e^\theta e^{i\theta}\right], \mathrm{Im}\left[e^\theta e^{i\theta}\right], \theta\right)$ and $\left(\mathrm{Re}\left[e^\theta e^{i\theta}\right], \mathrm{Im}\left[e^\theta e^{i\theta}\right], \ln e^\theta\right)$ are congruent and super-imposed in three dimensions. These curves instantiate $\left(\mathrm{Re}\left[re^{i\theta}\right], \mathrm{Im}\left[re^{i\theta}\right], f(r,\ \theta)\right]$ where $r = e^\theta$. Note that $Z = \ln e^\theta = \theta$, so the two curves are identical.

(iii) Nevertheless, use (20.4) to deduce that corresponding points of the two surfaces have the same curvature!

$$\mathcal{K}_1(r,\ \theta) = \frac{-1}{\left(r^2+1\right)^2} = \mathcal{K}_2(r,\ \theta).$$

Vasco (P. c., VDGF Forum) has pointed out that "corresponding points" means the two points above the (x, y)- or complex plane associated with $f_1(r,\ \theta)$ and $f_2(r,\ \theta)$ applied to the same (r, θ) pair. The points themselves are not generally the same point. The two corresponding points on their respective surfaces may be at different heights but locally the surfaces have the same curvatures.

$f_1(r,\ \theta) = \ln r$

$\partial_r(\ln r) = 1/r$

$\partial_r^2(\ln r) = -1/r^2$

$\partial_\theta(\ln r) = 0$

$\partial_\theta^2(\ln r) = 0$

$\partial_\theta\partial_r(\ln r) = 0$

$$\mathcal{K}_1 = \frac{r^2\left(-1/r^2\right)\left(0+r(1/r)-(0-0)^2\right.}{\left\{r^2\left[1+(1/r)^2\right]+0\right\}^2} = \frac{-1\times\left(0+1-(0-0)^2\right.}{\left\{r^2+1^2+0\right\}^2} = \frac{-1}{\left(r^2+1\right)^2}$$

$f_2(r,\ \theta) = \theta$

$\partial_r(\theta) = 0$

$\partial_r^2(\theta) = 0$

$\partial_\theta(\theta) = 1$

$\partial_\theta^2(\theta) = 0$

$\partial_\theta\partial_r(\theta) = 0$

$$\mathcal{K}_2 = \frac{r^2\cdot 0\cdot(0+r0)-(1-0)^2}{\left\{r^2\left[1+(0)^2\right]+(1)^2\right\}^2} = \frac{-1}{\left\{r^2+1\right\}^2}$$

The equations for $\mathcal{K}_1$ and $\mathcal{K}_2$ are the same, so $f_1(r,\ \theta)$ and $f_2(r,\ \theta)$ have the same curvature when r is the same.

Note: $\mathcal{K}$ depends only on r and goes to 0 as r $\longrightarrow \infty$. At $r = 0$, $\mathcal{K} = -1$. In the helicoid, it is hard to visualize two principle directions in which one has negative curvature and the other has positive curvature, but it must be the case since $\mathcal{K}_2 = \kappa_1\,\kappa_2 = -1$.