Needham, VDGAF, Chapter 21, Exercise 18, p. 224
Palmer, November 15 , 2023, 4:35 pm PT

18. Curvature Formula in Polar Coordinates. If $z = f(r, \theta)$, show that the curvature is given by this disappointingly complicated formula:

$$\mathcal{K} = \frac{r^2\, \partial_r^2 f\left(\partial_\theta^2 f + r\, \partial_r f\right) - (\partial_\theta f - r\, \partial_r \partial_\theta f)^2}{\left\{r^2\left[1 + (\partial_r f)^2\right] + (\partial_\theta f)^2\right\}^2} \qquad (20.4)$$

The small g in the denominator in the text should not be there. The following is adapted from the post by C. Y. Cheng at
https://math.stackexchange.com/questions/153371/how-do-i-compute-mean-curvature-in-cylindrical-coordinates

I doubt that my answer is what Needham had in mind, but it is the best I can do for the time being. I have altered Cheng's (Gauss') notation to better highlight the polar coordinates and ultimately I substituted Needham's notation to show that the result really is (20.4). I have indicated in red the places where my calculations differed from Cheng's. This derivation of (20.4) begins with Gauss's equation

$$K = \frac{\mathrm{eg} - f^2}{\mathrm{EG} - F^2},$$

where "e, f, g are the coefficients of the second fundamental form of patch x, and E, F, G are the coefficients of the first fundamental form."

$$S(r, \theta) = \boldsymbol{x}(r, \theta) = (r\cos\theta, r\sin\theta, h(r, \theta))$$

Let $\boldsymbol{x}_r$ and $\boldsymbol{x}_\theta$ be "the local surface tangent vectors at point p."

$$E = \|\boldsymbol{x}_r\|^2, \qquad F = \boldsymbol{x}_r \cdot \boldsymbol{x}_\theta, \qquad G = \|\boldsymbol{x}_\theta\|^2$$

The normalized vector at p is given by

$$\boldsymbol{U} = \frac{\boldsymbol{x}_r \times \boldsymbol{x}_\theta}{\|\boldsymbol{x}_r \times \boldsymbol{x}_\theta\|}$$

Then

$$S(r, \theta) = \boldsymbol{x}(r, \theta) = (r\cos\theta, r\sin\theta, h(r, \theta))$$

$$\boldsymbol{x}_r = (\cos\theta,\ \sin\theta,\ h_r)$$

$$\boldsymbol{x}_\theta = (-r\sin\theta,\ r\cos\theta,\ h_\theta)$$

$$\boldsymbol{x}_{rr} = (0,\ 0,\ h_{rr})$$

$$\boldsymbol{x}_{r\theta} = (-\sin\theta,\ \cos\theta,\ h_{r\theta})$$

$$\boldsymbol{x}_{\theta\theta} = (-r\cos\theta,\ -r\sin\theta,\ h_{\theta\theta})$$

Let $N = \sqrt{r^2\left(1 + h_r^2\right) + h_\theta^2}$. **U** is the unit normal.

$$\boldsymbol{U} = \frac{(h_r \sin\theta - r\,h_r\cos\theta, -h_\theta\cos\theta - r\,h_r\sin\theta, r)}{N}$$

$$E = \|\,\boldsymbol{x}_r\,\|^2 = 1 + h_r^2$$

$$G = \|\,\boldsymbol{x}_\theta\,\|^2 = r^2 + h_\theta^2$$

$$F = \boldsymbol{x}_r \bullet \boldsymbol{x}_\theta = h_r\,h_\theta$$

$$e = \boldsymbol{x}_{rr} \bullet \boldsymbol{U} = \frac{r\,h_{rr}}{N}$$

$$f = \boldsymbol{x}_{r\theta} \bullet \boldsymbol{U} = \frac{-h_\theta + r\,h_{r\theta}}{N}$$

$$g = \boldsymbol{x}_{\theta\theta} \bullet \boldsymbol{U} = \frac{r^2\,h_r + r\,h_{\theta\theta}}{N}$$

$$K = \frac{r^2\,h_{rr}(r\,h_r + h_{\theta\theta}) - (-h_\theta + r\,h_{r\theta})^2}{\left[r^2\left(1 + h_r^2\right) + h_\theta^2\right]^2} \qquad (1)$$

Let $h_r = \partial_r f$, $h_{rr} = \partial_r^2 f$, $h_\theta = \partial_\theta f$, $h_{\theta\theta} = \partial_\theta^2 f$, $h_{r\theta} = \partial_r \partial_\theta f$

Then substituting Needham's notation for the partial derivatives into (1),

$$K = \frac{r^2\,\partial_r^2\left(\partial_\theta^2 f + r\,\partial_r f\right) - (\partial_\theta f - r\,\partial_r \partial_\theta f)^2}{\left[r^2\left(1 + (\partial_r f)^2\right) + (\partial_\theta f)^2\right]^2}$$

which is what we wanted. I have checked some of the calculations with Mathematica. I believe a mistake in Cheng's calculation of the Gaussian "f" carried through into his calculation of K. If my calculations are correct, then (20.4) in Needham is correct.

It's possible that I have misinterpreted the exercise as calling for a derivation of (20.4). If one simply substitutes $\partial_r f$ for $\partial_x f$ and $\partial_\theta f$ for $\partial_y f$, where $\partial_r f = 0 = \partial_\theta f$ at (0,0) and while letting r = 1, the result is

$$\mathcal{K} = \partial_r^2 f\, \partial_r^2 f - (\partial_r \partial_\theta f)^2,$$

which is (20.3), and which shows that “the curvature [if $z = f(r, \theta)$] is given by this disappointingly complicated formula.”