Needham, VDGAF, Chapter 20, Exercise 17, p. 223
Palmer, June 24, 2023, 1:40 pm PST

17. *Use (20.2) and (20.3) to verify your conclusions in Exercise 7.*

(i) $z = exp\left(x^2 + 4\,y^2\right) - 1$

$$\mathcal{K} = \left(\partial_x^2 f\right)\left(\partial_y^2 f\right) - \left(\partial_x \partial_y f\right)^2 \qquad (20.3)$$

From extra calculations in exercise 7, we have

$$\left(\partial_x^2 f\right)(0) = 2\,e^{x^2+4\,y^2} + 4\,x^2\,e^{x^2+4\,y^2} = 2$$

$$\left(\partial_x^2 f\right) = 8\,e^{x^2+4\,y^2} + 64\,y^2\,e^{x^2+4\,y^2} = 8$$

$$\left(\left(\partial_x \partial_y f\right)(0)\right)^2 = 16\,\mathrm{xy}\,e^{x^2+4\,y^2} = 0$$

Using (20.3), $\mathcal{K} = 16 - 0^2 = 16$. This is the same result as in Ex. 7(i). Using (20.2), these values give a Shape Operator:

$$[S] = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$$

from which, using (15.7) we can deduce that the principle directions are $\kappa_1 = 2$, $\kappa_2 = 8$, which is the same result as in Ex. 7(i). We can also deduce that the shape near (0,0) is a hill or basin rather than a saddle.

(ii) $z = \ln\cos y - \ln\cos 2\,x$

From exercise 7, we have

$$\left(\partial_x^2 f\right)(0) = 4\sec^2 2\,x = 4$$

$$\left(\partial_y^2 f\right)(0) = -\sec^2 y = -1$$

$$\left(\left(\partial_x \partial_y f\right)(0)\right)^2 = (2\tan 2\,x)\,(-\tan y) = 0$$

Using (20.3), $\mathcal{K} = -4 - 0^2 = -4$. This is the same result as in Ex. 7(i). Using (20.2), these values give a Shape Operator:

$$[S] = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

from which, using (15.7) we can deduce that the principle directions are $\kappa_1 = 4$, $\kappa_2 = -1$, which is the same result as in Ex. 7(i). We can also deduce that the shape near (0,0) is a saddle rather than a hill or basin.