Needham, VDGAF, Chapter 21, Exercise 16, p. 222

Palmer, October 5 , 2023, 11:40 am PDT

16. Nonprincipal Coordinates*. Let us reanalyze the saddle surface, this time without first aligning the coordinate axes with the principal directions, as we did in the text.*

Out[∘]=

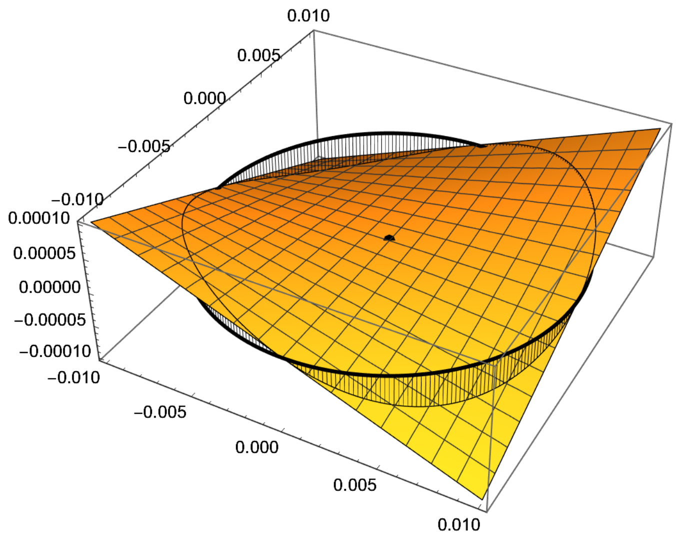


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Figure 1. Surface xy near origin
with heights from circle in (x,y)-plane
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(i) Sketch the surface $z = xy$ *near the origin (Hint: Recalling that* $\sin 2\theta = 2\sin\theta\cos\theta$*, find the height of the surface above and below the points of the origin-centred circle of radius r, for which* $x = r\cos\theta$ *and* $y = r\sin\theta$*.)*

$$height = x\,y = r^2 \sin\theta \cos\theta = \frac{1}{2} r^2 \sin 2\theta$$

Figure 1 plots the heights above and below the origin centred circle, r = 0.01.

Out[•]=

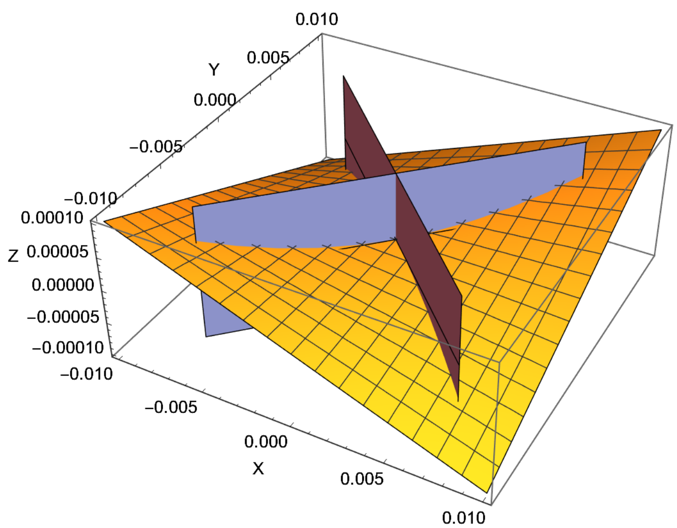

Figure 2. Planes of mirror symmetry on S = xy

(ii) Use (10.4), p. 111, to identify the principal directions.

(10.4) defines the principle directions as those defining mirror symmetry in the intersection with the tangent plane of two planes which also contain the surface normal. The principal directions in the tangent plane of the surface xy are $\theta_1 = \pi/4,\ \theta_2 = -\pi/4$. See Figure 2.

(iii) Use (20.2) to deduce that the Shape Operator has the matrix

$$[\mathcal{S}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Apply (20.2) to f(x,y) = xy.

$$[\mathcal{S}] = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_x \partial_y f & \partial_y^2 f \end{pmatrix} = \begin{pmatrix} \partial_x^2 (\mathrm{xy}) & \partial_x \partial_y (\mathrm{xy}) \\ \partial_x \partial_y (\mathrm{xy}) & \partial_y^2 (\mathrm{xy}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and deduce that $\mathcal{K} = -1$ and $\overline{\kappa} = 0$.

$$\mathcal{K} = \det[\mathcal{S}] = 0 \cdot 0 - 1 \cdot 1 = -1 \qquad \text{(Total curvature)}$$

$$\overline{\kappa} = \frac{1}{2} \nabla^2 f = \frac{1}{2} \left(\partial_x^2 (\mathrm{xy}) + \partial_y^2 (\mathrm{xy})\right) = 0 \qquad \text{(1/2 Trace[S])}$$

(iv) What is the geometric transformation $\mathcal{S}$ represented by [S]?

[S] applied to any point in the xy-plane is a reflection in the line y = x. Applied to a point p on a circle C centred at the origin, it sends p to the reflected point on C where C intersects with the chord orthogonal to the line y = x and passing through p.

(v) Use (iv) to determine the eigenvectors and eigenvalues of S, geometrically (not by calculation). (NOTE: The eigenvectors should point in the same direction you found in (ii).)

By inspection, the eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with eigenvalues 1 and -1. They can be multiplied by any real number. Apparently, one can create another [S] equivalent to that in (iii) and (viii) by writing:

$$[S] = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \text{ where } \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\mathrm{ad-bc}}}$$

(vi) Guided by either (ii) or (v), rotate the (x, y)-axes to obtain new (X, Y)-axes aligned with the principle directions, and show that the original equation now becomes $z = \frac{1}{2}\left(X^2 - Y^2\right)$.

Guided by (ii), we rotate the (x, y)- axes by $\theta = -\pi/4$. Assuming a point on a unit circle,

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$R_{-\pi/4} \circ \begin{pmatrix} x \\ y \end{pmatrix} = [R_{-\pi/4}].\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{-x+y}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \end{pmatrix}$$

Solve for x and y in terms of X and Y:

$$\mathrm{x} = \sqrt{2}\, X - y$$
$$\mathrm{y} = \sqrt{2}\, Y + x$$

$$x = \sqrt{2}\, X - \left(\sqrt{2}\, Y + x\right)$$

$$x = \frac{\sqrt{2}}{2}\,(X - Y)$$

Similarly

$$y = \frac{\sqrt{2}}{2}\,(X + Y)$$

Then it follows that

$$x\,y = \frac{1}{2}\left(X^2 - Y^2\right)$$

To obtain this result guided by (v), normalize [S]:

$$[S] = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

This is a rotation equivalent to $R_{-\pi/4}$ used when guided by (ii).

(vii) By comparing the new equation in (v) [?(vi)] *with (10.6), page 111, confirm the values of $\mathcal{K}$ and $\overline{\kappa}$ you found in (iii).*

The equation on p. 111 is

$$z = \frac{1}{2}\,\kappa_1\,x^2 + \frac{1}{2}\,\kappa_2\,y^2$$

$$z = \mathrm{xy} \equiv \frac{1}{2}\,\kappa_1\,X^2 + \frac{1}{2}\,\kappa_2\,Y^2 = \frac{1}{2}\left(X^2 - Y^2\right)$$

The result is correct if we let $\kappa_1 = 1$ and $\kappa_2 = -1$.

...confirm the values of $\mathcal{K}$ and $\overline{\kappa}$.

$$\mathcal{K} = \kappa_1\,\kappa_2 = -1$$

$$\overline{k} = \frac{1}{2}\,\nabla f = \frac{1}{2}\left(\partial_X^2 + \partial_Y^2\right) = \frac{1}{2} \times (1 - 1) = 0$$

(viii) Use (20.2) to write down [S] in the new (X, Y)-coordinate system of (vi), and confirm that $\mathcal{K}$ and $\overline{\kappa}$ have not changed.

$$[S] = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_x \partial_y f & \partial_y^2 f \end{pmatrix}$$

$$\partial_X^2 f = \partial_X^2\left(\frac{1}{2}\left(X^2 - Y^2\right)\right) = 1$$

$$\partial_X \partial_Y f = \partial_X \partial_Y \frac{1}{2}\left(X^2 - Y^2\right) = 0$$

$\partial_Y^2 f = -1$

$[S] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathcal{K} = \kappa_1 \, \kappa_2 = -1$

$\overline{k} = \frac{1}{2} \nabla f = \frac{1}{2} \left(\partial_X^2 + \partial_Y^2\right) = \frac{1}{2} \times (1 - 1) = 0.$

This confirms that $\mathcal{K}$ and $\overline{k}$ have not changed with the rotation of axes.

(ix) Verify that the two different matrices [S] in (iii) and (viii) (representing the same linear transformation S) conform to the general matrix formula (15.20), page 161.

$$[S] = \begin{pmatrix} \kappa(\boldsymbol{E}_1) & \frac{\triangle\kappa}{2} \sin 2\theta \\ \frac{\triangle\kappa}{2} \sin 2\theta & \kappa(\boldsymbol{E}_2) \end{pmatrix} \qquad (15.20)$$

$$[S] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(iii)}$$

$$[S] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(viii)}$$

To verify that [S] conforms to (15.20), calculate the entries of (15.20) using Euler’s formula (pp. 110, 160) for the diagonals and $\frac{\triangle\kappa}{2} \sin 2\theta$ for the off-diagonals.

Does [*S*] in (iii) conform to (15.20)?

From (iii) we know that $\overline{\kappa} = \frac{\kappa_1 + \kappa_2}{2} = 0$ for [S] in (iii). [See (12.1), p. 130]. Hence $\kappa_1 = -\kappa_2$.

From (10.1), $\triangle\kappa = \kappa_1 - \kappa_2 = 2\,\kappa_1$. This means that the off-diagonals are

$$\kappa_1 \sin 2\theta = \kappa_1 \sin(2\,(-\pi/4)) = -\kappa_1 \sin(\pi/2) = -\kappa_1$$

This is a conformal value for the off-diagonals because the determinant is $\kappa_1 \, \kappa_2 = -(-\kappa_1)^2 = -1$, assuming the diagonals are zero. So we need only show that the diagonals are both zero. Use Euler’s Curvature Formula, p. 160).

$$\kappa(E_1) = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha = 1 \cdot \cos^2(-\pi/4) - 1 \cdot \sin^2(-\pi/4)$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\kappa(E_2) = \kappa_1 \sin^2\alpha + \kappa_2 \cos^2\alpha = 1 \cdot \sin^2(-\pi/4) - 1 \cdot \cos^2(-\pi/4)$$

$$= \left(-\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$$

We conclude that [S] in (iii) conforms to (15.20).

Does [S] in (viii) conform to (15.20)?

For (viii), there is no twist rotation away from the principal directions, because $[M]^T = [M]$, so θ (or α) = 0. Then

$$\kappa(E_1) = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha = 1 \cdot \cos^2 \times 0 - 1 \cdot \sin^2 \times 0 = 1 - 0 = 1$$

$$\kappa(E_2) = \kappa_1 \sin^2 0 + \kappa_2 \cos^2 0 = 1 \cdot 0 - 1 \cdot \cos^2 \times 0 = -1$$

The diagonals conform to (15.20). Now check the off-diagonals.

$$\frac{\Delta\kappa}{2} \sin 2\theta = 2\,\kappa_1 \sin(2 \cdot 0) = 0$$

The off-diagonals are zero. We conclude that [S] in (viii) conforms to (15.20).