Needham, VDGAF, Chapter 20, Exercise 15, p. 222
Palmer, November 4, 2023,10 am PT

***15. Cartesian Formulas for the Shape Operator, Curvature, and Mean Curvature**. Let T_p be the tangent plane of a general surface $\mathcal{S}$ at a point p. Introduce Cartesian coordinates (x, y) into T_p, and take p = (0, 0), so that the normal **n** there points along the z-axis. Then, locally, $\mathcal{S}$ is given by*

$$z = f(x,\ y) \qquad \text{where } f(0,\ 0) = 0 \quad \text{and } \partial_x f = 0 = \partial_y f \quad \text{at } (0,0).$$

(i) Note that if we define $F(x,\ y,\ z) = f(x,\ y) - z$, then F is constant on $\mathcal{S}$. Deduce that ∇F is normal to $\mathcal{S}$, and therefore

$$\boldsymbol{n} = \frac{1}{\sqrt{1+(\partial_x f)^2+(\partial_y f)^2}}\begin{pmatrix} \partial_x f \\ \partial_y f \\ -1 \end{pmatrix}.$$

Something doesn't add up. Since $z = f(x,\ y)$, it should always be the case that $F(x,\ y,\ z) = f(x,\ y) - z = 0$, so F is a constant 0 on $\mathcal{S}$. Still, it is hard to see why ∇F would be orthogonal. Vasco wrote:

If ∇F is not orthogonal to the surface $\mathcal{S}$ then it has a component in some direction on $\mathcal{S}$ which would mean that F was not constant on the surface because the component of ∇F represents a change in F with respect to movement on the surface. (P. C. VDGF Forum)

Since $F(x,\ y,\ z) = f(x,\ y) - z$, one can write

$$\nabla F = \begin{pmatrix} \partial_x F \\ \partial_y F \\ \partial_z F \end{pmatrix} = \begin{pmatrix} \partial_x f \\ \partial_y f \\ -1 \end{pmatrix}$$

Then, since we know that ∇F is normal to $\mathcal{S}$ and **n** is the **unit** normal,

$$\boldsymbol{n} = \frac{\nabla F}{|\nabla F|} = \left(1\Big/\left(\sqrt{1+(\partial_x f)^2+\left(\partial_y f\right)^2}\right)\right)\begin{pmatrix} \partial_x f \\ \partial_y f \\ -1 \end{pmatrix}$$

(ii) Deduce that the matrix of the Shape Operator at (0,0) is

$$[S] = \begin{pmatrix} \partial_x^2 f & \partial_x \partial_y f \\ \partial_x \partial_y f & \partial_y^2 f \end{pmatrix}, \qquad (20.2)$$

from which it follows immediately that

$$\mathcal{K} = \kappa_1 \kappa_2 = det[S] = (\partial_x^{\,2} f)(\partial_y^{\,2} f) - (\partial_x \partial_y f)^2 \qquad (20.3)$$

In this exercise, the Shape Operator lies in a plane parallel to the xy-plane, so it must be equivalent to the sum of the partial derivatives of ▽F in the xy-plane at the origin. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be unit vectors in the x and y directions respectively, as in (15.6). (Or if we prefer, we can calculate from part (i) that the normal goes to 1 and multiply it through.)

$$S(\boldsymbol{e}_1) \equiv \partial_x \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} \partial_x^2 f \\ \partial_x \partial_y f \end{pmatrix}$$

$$S(\boldsymbol{e}_2) \equiv \partial_y \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} \partial_y \partial_x f \\ \partial_y^2 f \end{pmatrix}$$

$$[S] = [S(\boldsymbol{e}_1),\ S(\boldsymbol{e}_2)] \equiv \begin{pmatrix} \partial_x^2 f & \partial_y \partial_x f \\ \partial_x \partial_y & \partial_y^2 f \end{pmatrix}$$

This is the answer in (20.2). Then (20.3) is just the determinant (expansion factor) of [S]. The derivation of $\overline{\kappa}$ is given in the exercise. This answer to part (ii) was influenced by a very similar exercise in Barrett O'Neill, *Elementary Differential Geometry* (2nd Ed), 1997, Elsevier, p. 200. Vasco eliminates the problem with the negative sign by defining $F = z - f(x,\ y)$. By using (15.6) we can avoid a commitment regarding sign.