Needham, VDGAF, Chapter 21, Exercise 14, p. 222
Palmer, Oct 30, 2023, 4:30 pm PDT

Out[49]=

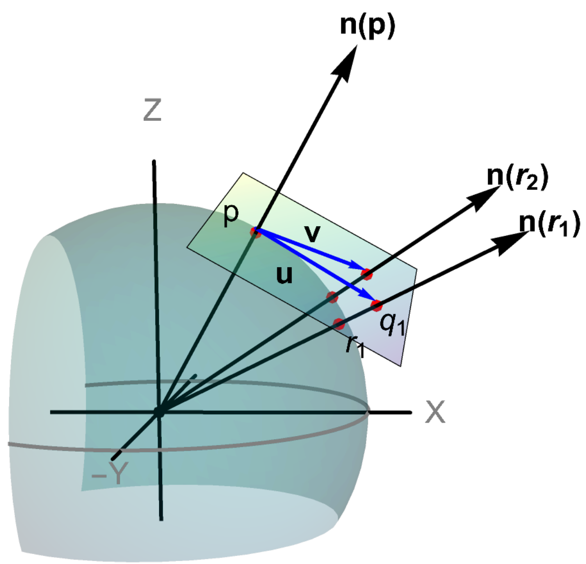

Figure 1. Shape operators on spherical surface. Cf. [15.1], [15.2].

14. ***Gaussian Curvature in terms of the Shape Operator****. If* **u** *and* **v** *are short tangent vectors at a point p on a surface with shape Operator S, show geometrically that* $S(\boldsymbol{u}) \times S(\boldsymbol{v}) = \mathcal{K}(p)[\boldsymbol{u} \times \boldsymbol{v}]$*....*

Geometrically, S(**u**) and S(**v**) are the directional derivatives $-\nabla_{\boldsymbol{u}}\, \boldsymbol{n}$ and $-\nabla_{\boldsymbol{v}}\, \boldsymbol{n}$, which means they approximate the small vector $\delta\mathbf{n}/\epsilon$ on each of **u** and **v,** where ϵ is the length of a movement (ultimately) in the surface of the sphere that follows the direction of **u** or **v**. See [15.1] and [15.2], p. 150. S(**u**)×S(**v**) can be taken as the area of a parallelogram formed by S(**u**) and S(**v**) where the diagonal is their sum. Since I have chosen to illustrate the problem with a unit sphere, one can re-anchor the normals $\boldsymbol{n}_p$, $\boldsymbol{n}_1$, and $\boldsymbol{n}_2$ from p, r_1, and r_2 to the origin, where the vectors **u** and **v** can now do double duty as S(**u**) and S(**v**) on the spherical (Gauss) map (p. 131). So we have superimposed the spherical map of the normals onto the surface sphere under investigation. The tangent plane is mapped to itself in place. Thus, in this example $S(\boldsymbol{u}) \times S(\boldsymbol{v}) = \boldsymbol{u} \times \boldsymbol{v}$.

The notation $\mathcal{K}$(p)[**u**×**v**] appears to describe the product of the Gaussian intrinsic curvature multiplied by the area of the parallelogram obtained with the cross product of the small vectors **u** and **v**. But if one takes this $\mathcal{K}$(p) to be extrinsic curvature $\mathcal{K}_{\text{ext}}(p)$, the approximation formula above (12.8) applies:

$$\delta\tilde{\mathcal{A}} \asymp (\kappa_1\,\kappa_2)\,\delta\mathcal{A}$$

or

$$\delta\tilde{\mathcal{A}} \asymp \mathcal{K}\,\delta\mathcal{A}$$

One can see from Figure 1 and the considerations in the previous paragraph that it is now possible to write

$$S(\boldsymbol{u}) \times S(\boldsymbol{v}) = \mathcal{K}(p)[\boldsymbol{u} \times \boldsymbol{v}]$$

This result applies as well to non-spherical surfaces. The use of the sphere for both the surface and the spherical map is only a device to simplify the illustration. Here, since I have chosen a unit sphere, the curvature is $\mathcal{K} = 1/R^2 = 1/1 = 1$, so $S(\boldsymbol{u}) \times S(\boldsymbol{v}) = [\boldsymbol{u} \times \boldsymbol{v}]$.