Needham, VDGAF, Chapter 21, Exercise 13, p. 222

Palmer, April 16, 2023, 7 pm PDT

13. Vanishing Shape Operator ⟺ Flat*. Show, both geometrically and by calculation, that if the Shape Operator vanishes identically, then the surface is a portion of a plane.*

Geometrically, consider the change in a normal **n**(p) to surface $\mathbb{S}$ that results from a movement from point p to point r in the direction of a vector **v** in the tangent plane T_p. If the Shape Operator vanishes identically (i.e. for all movements on $\mathbb{S}$), then the change in **n** from **n**(p) to **n**(r) both re-anchored at the origin of a sphere) is zero for all values of p and r. This can only happen if **n** remains normal (parallel to the Z axis in the sphere), which means that $\mathbb{S}$ is a plane. (See [15.1] and [15.2], p. 150.)

By calculation: Two vectors emanating from a point p in a tangent plane T_p to a surface $\mathbb{S}$ at any angle in the plane except 0 or π define the tangent plane . If the shape angle vanishes identically on $\mathbb{S}$, then we may assume that it vanishes for the sum of any two such vectors **v** and **w** at any point p. Let a and b be non-zero arbitrary constants.

$$S(a\boldsymbol{v} + b\boldsymbol{w}) = 0 \Longrightarrow \mathrm{aS}(\boldsymbol{v}) + \mathrm{bS}(\boldsymbol{w}) = 0$$

$$\Longrightarrow \mathrm{aS}(\boldsymbol{v}) = 0 \textbf{ and } \mathrm{bS}(\boldsymbol{w}) = 0$$

Since the Shape Operator is identically zero everywhere, it is zero on $\mathbb{S}$ in all directions in T_p. Therefore, the surface is a plane which is in fact identical to T_p.