Needham, VDGAF, Chapter 21, Exercise 1, p. 219.
Palmer, November 19, 2022, 8:20 am PT

1. Computational Proof of the Curvature Formula. Figure [8.7] provided a geometrical proof of (8.7). Prove this instead by calculation. (Hint: If the curve [x(t), y(t)] is traced at unit speed, then $\dot{x} = \cos\varphi$*, and* $\dot{y} = \sin\varphi$*.)*

We want to calculate $\kappa = \ddot{y}/\dot{x}$. From (8.6),

$$\kappa = \frac{\ddot{y}\,\dot{x} - \ddot{x}\,\dot{y}}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}$$

Since the curve is traced at unit speed, this can be rewritten substituting $\dot{x}^2 + \dot{y}^2 = 1$.

$$\kappa = \ddot{y}\,\dot{x} - \ddot{x}\,\dot{y} = \ddot{y}\cos\varphi - \ddot{x}\sin\varphi$$

$$= \cos^2\varphi\,\dot{\varphi} + \sin^2\varphi\,\dot{\varphi} = \dot{\varphi}$$

$$\ddot{y}/\dot{x} = \cos\varphi\,\dot{\varphi}/\cos\varphi = \dot{\varphi}$$

So one can write $\kappa = \dot{\varphi} = \ddot{y}/\dot{x}$.