Notes on Needham, Chapter 11

G. Palmer, October 6, 2024

P. 116. *[exercise] ... The acceleration of the projection equals the projection of the acceleration*.

Assume a particle traced at constant speed (p. 116, ¶. 3). From (11.1) and [11.2], where gamma γ is the angle at p between the plane of C and T_p, and gamma $\gamma = \alpha$,

$$k_g \equiv \tilde{\kappa} \asymp \kappa \cos \gamma = \kappa \cos \alpha$$

κ_g is the magnitude of the geodesic curvature vector, so it is the curvature of a circle in T_p and it can be called the *geodesic acceleration vector* (p. 116, ¶. 3).

By the definition of curvature on a plane curve at p, the tangency of T_p to the surface in [11.2], and the right angle at p, one can deduce that the vector $\boldsymbol{\kappa}_g$ lies on the unit vector $\frac{-\boldsymbol{T}\times\boldsymbol{n}}{|\boldsymbol{T}\times\boldsymbol{n}|}$, which is perpendicular to Π_T at p and lies in T_p. Then

$$\boldsymbol{\kappa}_g = \kappa_g \frac{-\boldsymbol{T}\times\boldsymbol{n}}{|\boldsymbol{T}\times\boldsymbol{n}} = k \cos \gamma \left(\frac{-\boldsymbol{T}\times\boldsymbol{n}}{|\boldsymbol{T}\times\boldsymbol{n}|}\right)$$

The projection of the acceleration vector $\boldsymbol{\kappa}$ onto T_p is

$$\operatorname{proj}_{T_p}(\boldsymbol{\kappa}) = k \cos \gamma \left(\frac{-\boldsymbol{T}\times\boldsymbol{n}}{|\boldsymbol{T}\times\boldsymbol{n}|}\right) = \boldsymbol{\kappa}_g$$

Since the two are the same, we can say that *the acceleration of the projection equals the projection of the acceleration*.

In my opinion, the two diagrams [11.1] and [11.2] could be more consistent by using α or γ instead of both for what seems to be the corresponding angle. Note that the normal vector **n** in[11.2] is not the same as the normal **N** of the Frenet frame, which is not shown, but which has the same direction as $\boldsymbol{\kappa} = \kappa \boldsymbol{N}$ for a constant speed curve. The vector **n** is normal to the surface $\mathcal{S}$, not to the plane curve C. Hence, we see the binormal $\boldsymbol{B}$ orthogonal to the plane $\mathbf{T}\boldsymbol{\kappa}$, not orthogonal to **n**, but at the angle γ from **n** and Π_T.

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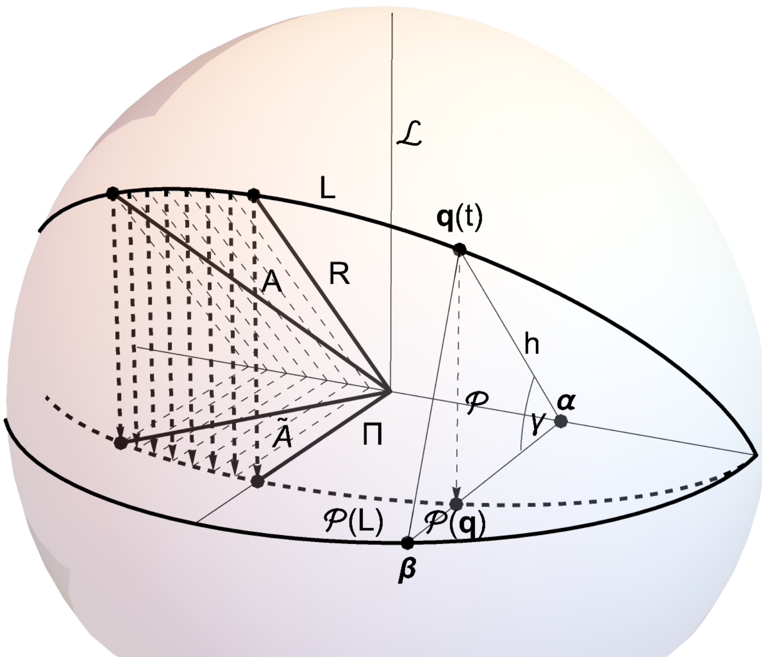

Figure 1. Projection of A to $\tilde{A}$

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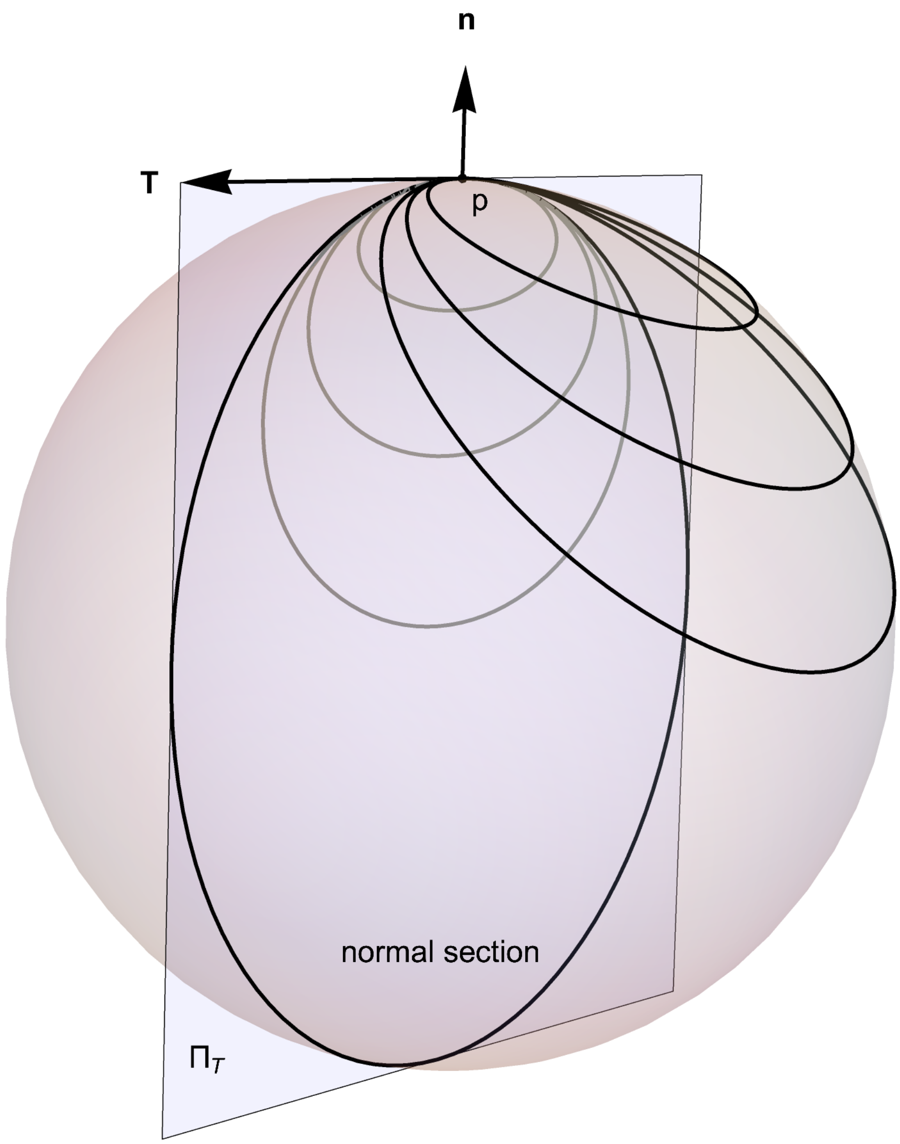

Figure 6. Circles centered at various
γ angles from **n** and projected onto Π_T

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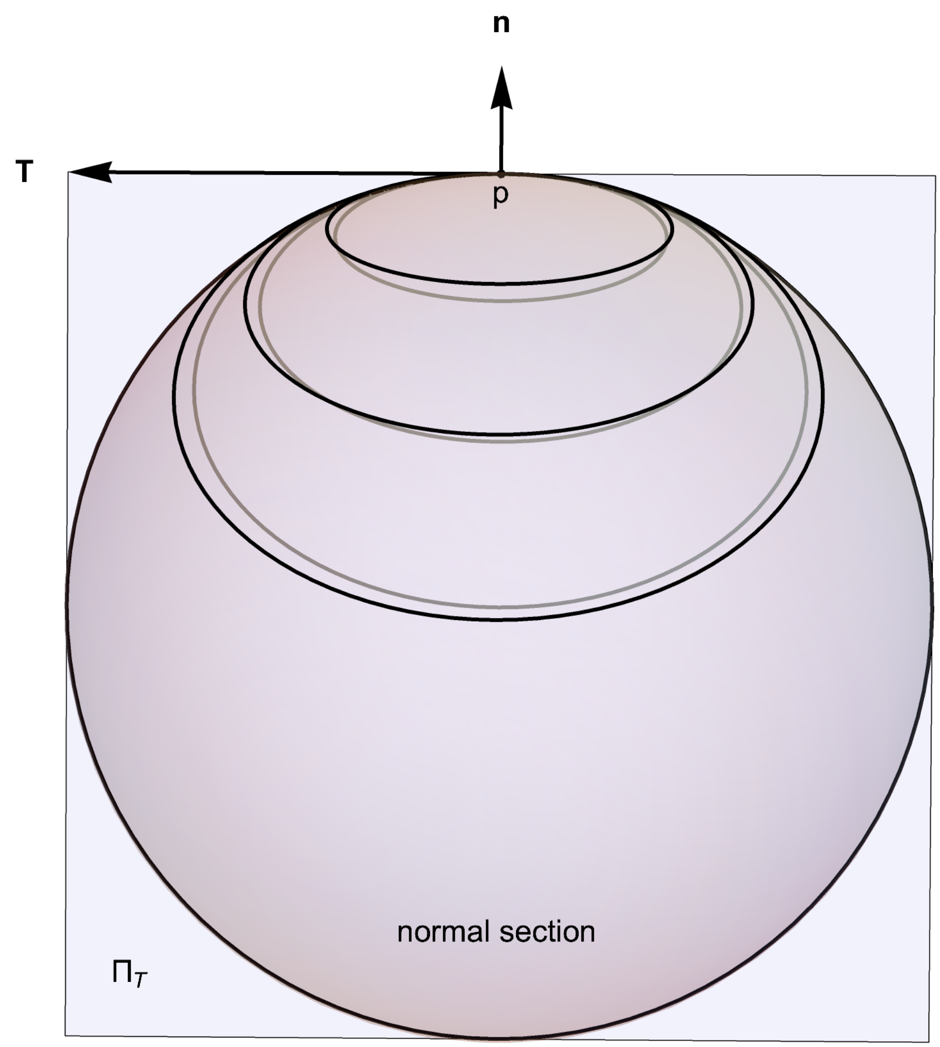

Figure 7. Side view of circles projected onto Π_T

P. 117, §11.2, ¶2. *All these curves locally have the same projection onto Π_T: near p, these projections all look like the normal section, which in turn looks like [the arc of] a circle within $\Pi_{\boldsymbol{T}}$ of radius $(1/\kappa_{\boldsymbol{n}})$ centred at $(1/\kappa_{\boldsymbol{n}})\,\boldsymbol{n}$.*

Figures 6 and 7 appear out of order because this section was inserted months after the other sections.

Figure 7 shows that the arcs of the projected circles onto the normal section can only look like the arcs of circles near p. Otherwise, they resemble the arcs of ellipses.

P. 122. [exercise], ¶ 2. *It is easy to see geometrically that if the plane of L makes an angle γ with Π, then* $\det \mathcal{P} = \cos\gamma$.

As shown in Figure 1, h is the distance of $\mathbf{q}$(t) from the x-axis. The angle $\boldsymbol{q}(t) - \mathcal{P}(\boldsymbol{q}) - \boldsymbol{\alpha}$ is a right angle. The hypotenuse h is therefore contracted by $\mathcal{P}$ to $h\cos\gamma$. At the apex of L, h = R. Consider the two triangles T1 = $\boldsymbol{\alpha} - \boldsymbol{\beta} - \boldsymbol{q}(t)$ and T2 = $\boldsymbol{\alpha} - \mathcal{P}(\boldsymbol{q}) - \boldsymbol{q}(t)$. Both triangles have height hgt = $|\,\boldsymbol{q}(t) - \mathcal{P}(\boldsymbol{q})\,|$.

$$\mathcal{A}(\mathrm{T1}) = \tfrac{1}{2} R \cdot \mathrm{hgt} \quad \text{(the larger triangle)}$$

$$\mathcal{A}(\mathrm{T2}) = \tfrac{1}{2} \left| \mathcal{P}(\boldsymbol{q}) - \boldsymbol{\alpha} \right| \cdot \mathrm{hgt} \quad \text{(the smaller triangle)}$$

$$\frac{\mathcal{A}(\mathrm{T2})}{\mathcal{A}(\mathrm{T1})} = \frac{|\mathcal{P}(\boldsymbol{q}) - \boldsymbol{\alpha}|}{R} = \frac{R\cos\gamma}{R} = \cos\gamma$$

Hence, we can say that $\mathcal{P}$ contracts the area of T1 by a factor of $\cos\gamma$, so we can write $\det\mathcal{P} = \cos\gamma$.

Can we use this information to deduce the area of the projection of A onto $\tilde{A}$ as in [11.7]? Since A and $\tilde{A}$ lie in the planes of the geodesic L and the xy-plane Π, respectively, it can also be deduced that $\tilde{A} = \cos\gamma\, A$. Since projection to Π subjects every point of A to contraction of its distance from the x-axis by the constant $\cos\gamma$, the area of A projected to Π is contracted by the same amount to become $\tilde{A}$. One could illustrate this with a near infinite number of rectangles or lines of infinitesimal width perpendicular to the x-axis, with segments intersecting and lying in A, extending only to the boundaries of A so as to fill the triangle. All such lines are projected to area Ã where they are contracted by $\cos\gamma$. Integrating all such contracted lines within the boundaries of A results in the contracted area Ã. Hence $\tilde{A} = \cos\gamma\, A$.

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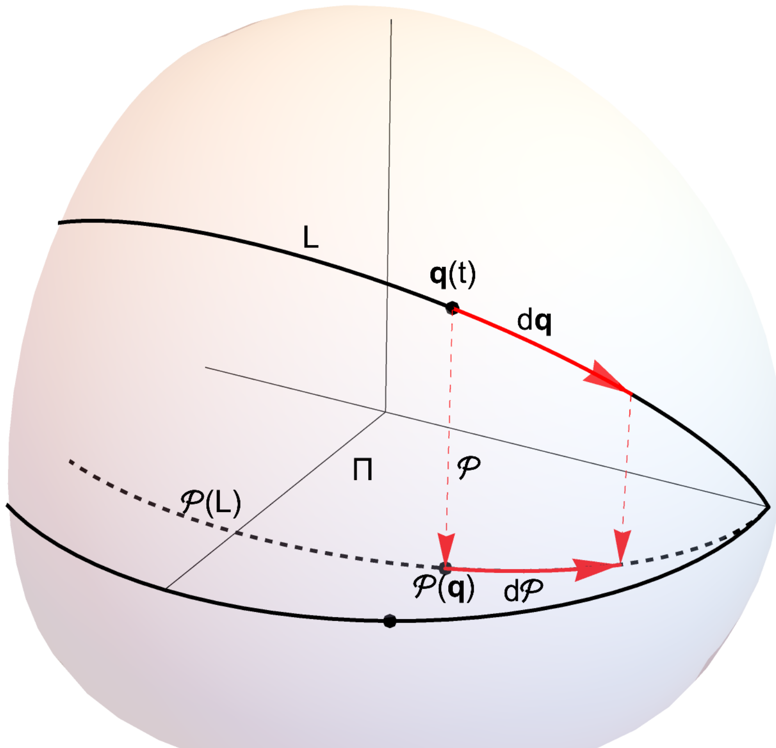

Figure 2. d𝒫[q]/dt = 𝒫[dq/dt]

P. 122. [exercise], ¶ 4. *If* $\boldsymbol{q}(t)$ *is any trajectory in space, the velocity of the orthogonal projection* $\mathcal{P}(\boldsymbol{q})$ *onto a plane* Π *is the projection of the velocity* $\frac{d}{dt}\mathcal{P}[\boldsymbol{q}] = \mathcal{P}\left[\frac{d\boldsymbol{q}}{dt}\right]$. (11.7)

Vasco (Forum, p.c.) wrote: *Since P is a linear transformation it follows that* $\mathcal{P}(\boldsymbol{q}) = P\boldsymbol{q}$ *where P is a constant matrix.* Then

$$\frac{d}{dt}\mathcal{P}[\boldsymbol{q}] = \frac{d}{dt}[P\cdot\boldsymbol{q}] = P\cdot\frac{d\boldsymbol{q}}{\mathrm{dt}} = \mathcal{P}\left[\frac{d\boldsymbol{q}}{\mathrm{dt}}\right].$$

Alternatively, one need only let the angle between the plane of q(t) and the plane Π go to zero to see in Figure 2 that $\mathcal{P}[d\boldsymbol{q}] = d\mathcal{P}[\boldsymbol{q}]$. Since the infinitesimals, here immensely exaggerated, both occur in the same time dt, we write

$$\frac{d}{dt}\mathcal{P}[\boldsymbol{q}] = \mathcal{P}\left[\frac{d\boldsymbol{q}}{dt}\right].$$

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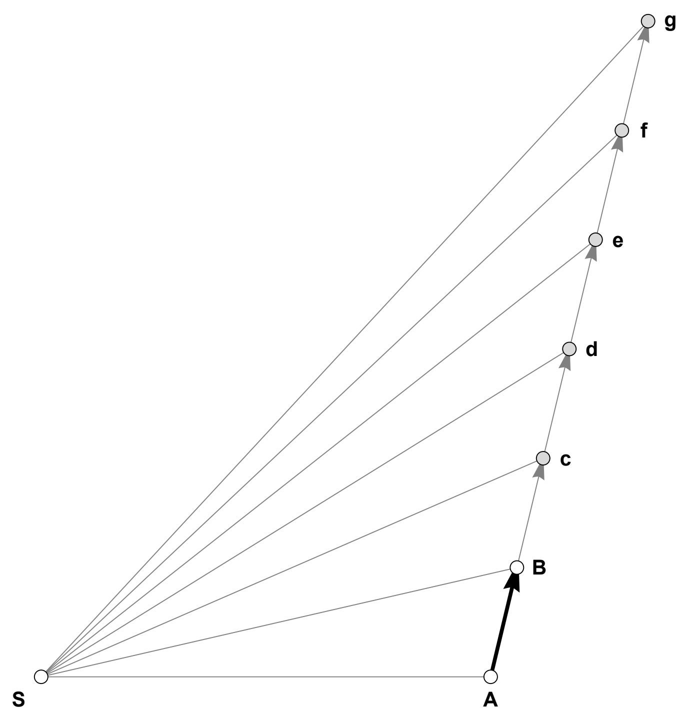

Figure 3. Sweeping proportional areas in proportional times; See [11.6], VDGF

P. 126. *Lastly observe [exercise] that the converse is true; this is proposition 2 of the Principia. Every body that moves in some curved line described in a plane and, by a radius drawn to a point describes areas around that point proportional to the times, is urged by a centripetal force tending towards the same point.*

Needham (p. 123) stated that the sweeping of equal areas in equal times is a universal property of all central force fields. Since a body in motion tends to remain in motion in the same direction, we can plot the free motion of a body at **A** set moving at a constant speed in the direction of **B** progressing as far as **f** (Figure 3). Each labeled point on the right side marks a distance equal to $|\boldsymbol{B}-\boldsymbol{A}|$. We know from the proof-by-shearing in 11.7.3 that the 6 triangles have equal areas swept out over equal times. Each new triangle in the series can be sheared to match the previous. It is clear from the plot, that they sweep different angles. If one were to plot an arbitrary curve C (not shown) across the rays and connect points of intersection to **S**, one could deduce that it would be necessary to apply centripetal force to a body on the **A** to **f** trajectory to make it conform to C. With no centripetal force (no central force field), the total swept out area would never exceed the boundary of the y-axis, so the body would not bend around **S**. With appropriate components of centripetal force, the body would bend back to C.

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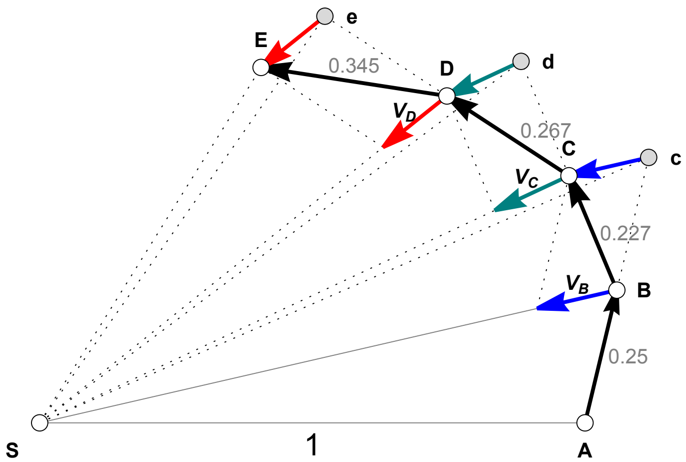

Figure 4. Sweeping equal areas in equal times; See [11.6]

The Construction

In Figure 4, taps are shown with colored arrows and labeled $\boldsymbol{V_i}$, where subscript **i** indicates the point where the tap was applied.

Note that the time required for the body in Figure 4 to traverse a segment is equal for segments ($\boldsymbol{B} - \boldsymbol{A}$), ($\boldsymbol{C} - \boldsymbol{B}$), ($\boldsymbol{D} - \boldsymbol{C}$), and ($\boldsymbol{E} - \boldsymbol{D}$) . The black arrows indicate indicate velocities. Both speeds and directions vary. Speeds are indicated with light gray labels.

1. Plot an arrow similar to ($\boldsymbol{B} - \boldsymbol{A}$) in [11.6]. Let this arrow have arbitrary length and arbitrary direction.
2. Plot an extension of ($\boldsymbol{B} - \boldsymbol{A}$) which is just ($\boldsymbol{B} - \boldsymbol{A}$) anchored at **B**. Call the point of this arrow **c**.
3. Set an arbitrary constant magnitude β for the "tap".
4. Apply the tap to **B** to obtain the vector $\boldsymbol{V_B}$, which has magnitude β and direction of $\boldsymbol{B}$: $\boldsymbol{V_B} = \beta e^{i\arg(\boldsymbol{B})}$.
5. Plot $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{T_B}$. (Alternatively, plot ($\boldsymbol{B} - \boldsymbol{V_B}$) + ($\boldsymbol{B} - \boldsymbol{A}$).)
6. Plot a third arrow equal to ($\boldsymbol{C} - \boldsymbol{B}$) anchored at **C**. Call the point of this arrow **d**.
7. Apply the tap to **C** to obtain $\boldsymbol{V_C} = \beta e^{i\arg(\boldsymbol{C})}$.
8. Plot $\boldsymbol{D} = \boldsymbol{d} - \boldsymbol{V_C}$. (Alternatively, plot ($\boldsymbol{C} - \boldsymbol{V_C}$) + ($\boldsymbol{c} - \boldsymbol{B}$).)
9. Apply the tap to **D** to obtain $\boldsymbol{V_D} = \beta e^{i\arg(\boldsymbol{D})}$.
10. Plot $\boldsymbol{E} = \boldsymbol{e} - \boldsymbol{V_D}$. (Alternatively, plot ($\boldsymbol{D} - \boldsymbol{V_D}$) + ($\boldsymbol{d} - \boldsymbol{C}$).)

This construction demonstrates the sweeping of equal areas in equal times.

How do we know the times are equal? To give an example, the time for **C** to **d** equals that for **B** to **C** by construction, because Cd is just a copy of BC. By the vector diagram, the time for traversal of **C** to **D** equals that of **C** to **d** and therefore, of **B** to **C**.

How do we know that the areas are equal beyond the obvious equality of $\mathcal{A}$(SBc) to $\mathcal{A}$(SAB)? The argument by shear is valid for all subsequent triangles, but Vasco (P. c., VDGF forum) has provided a simple geometric argument. The area of SCc equals that of BCc by sharing a base and equal heights (possibly hard to discover in the given construction). Then $\mathcal{A}$(SBc) + $\mathcal{A}$(SCc) = $\mathcal{A}$(SBC) + $\mathcal{A}$(BCc). Since both sides sum to the same quadrilateral, $\mathcal{A}$(SBC) = $\mathcal{A}$(SBc) . This reasoning can be applied at all the points **A, B, C, D, E**, ...

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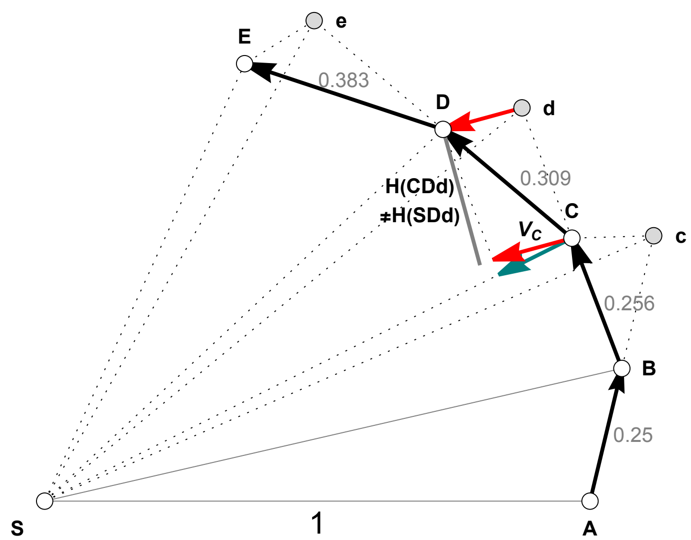

Figure 5. Red taps not centripetal

Now one can present a better argument that centripetal taps are required. Suppose a tap varies from the centripetal, as with $\boldsymbol{V_C}$ in Figure 5. The solid gray line labeled "≠H(CDd)" is perpendicular to $\boldsymbol{V_c}$ but it does not give the height of SDd. Hence one cannot be sure in general that $\mathcal{A}$(SDd) = $\mathcal{A}$(DCd), or that $\mathcal{A}$ (SCD) = $\mathcal{A}$ (SCd). Since equal areas are not swept out in equal times, it must be the case that centripetal taps are required as shown in Figure 4.